On the Linear Ranking Problem for Integer Linear-Constraint Loops

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February 10, 2020

Problem statement

Consider a loop described by linear constraints

```
while $x_2 - x_1 \leq 0, x_1 + x_2 \geq 1$ do
    $x_2' = x_2 - 2x_1 + 1$
    $x_1' = x_1$
end while
```

Question: is there a linear ranking function for this loop?

- Proves termination.
- Provides an upper bound on the number of iterations.

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    $x_1' = x_1$
**end while**
```

Is there a LRF for this loop over rational variables?

Problem known to be decidable in polynomial time. In this case, there can't be a LRF since loop doesn't terminate for a starting point $(\frac{1}{2}, \frac{1}{2})$.

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What if we restrict loop variables to integers?

There is a LRF $\rho(x) = x_1 + x_2 - 1$. Applying the algorithm for a rational case is not complete for an integer loop.

Problem statement

The paper presents answers to three key questions.

- Is there a complete algorithm for deciding the existence of LRF for an integer loop?
- Is this decidable in polynomial time?
- When does this problem have a polynomial solution?

Outline

1. Preliminaries
   - Integer polyhedra
   - Loop notation
   - Linear Ranking Function
2. Linear Ranking Problem
   - Hardness
   - Inclusion in coNP
     - Single path loops
     - Multipath loops
3. Polynomial-time cases
   - Integral polyhedron
   - Two-variable inequalities
   - Octagonal relations
   - Strongly polynomial
4. Summary

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Polyhedron

Rational polyhedron

The set of solutions to a system of inequalities $Ax \leq b$

$$P = \{x \in \mathbb{Q}^n \mid Ax \leq b\}$$

where $A \in \mathbb{Q}^{m \times n}$, $x \in \mathbb{Q}^n$ a $b \in \mathbb{Q}^m$.

Recession directions

The set of solutions to a system of inequalities $Ay \leq 0$, denoted as $\mathcal{R}_P$.

Integer hull

The integer hull P_I of polyhedron P is a set of all convex combinations of points in $I(P) = P \cap \mathbb{Z}$ i.e. integer points in P.

A polyhedron P is called an *integer polyhedron* or *integral* iff $P = P_I$.

Generator representation

From the Minkowski-Weyl Theorem, we know that every polyhedron has a representation as a Minkowski sum of a convex hull and a cone.

$$P = \text{convhull}(x_1, \ldots, x_m) \; + \; \text{cone}(y_1, \ldots, y_t)$$

where y_j are recession directions of P. As a consequence, for each $x \in P$ exist rationals $a_i, b_j \geq 0$ such that

$$x = \sum_{i=1}^{m} a_i \cdot \mathbf{x}_i + \sum_{j=1}^{n} b_j \cdot \mathbf{y}_j$$

where $\sum_{i=1}^{m} a_i = 1$. For an integral polyhedron, there is a representation where vector $\mathbf{x}_i, \mathbf{y}_j$ are integer.

Single-path loop

Single-path linear-constraint loop

An *SLC* loop over variables $x_1, \dots, x_n$ is given in the form

$$\text{while}(B\mathbf{x} \leq \mathbf{b}) \text{ do } A \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}$$

where $\mathbf{x} = (x_1, \dots, x_n)^T$, $\mathbf{x}' = (x'_1, \dots, x'_n)^T$, $B \in \mathbb{Q}^{p \times n}$, $A \in \mathbb{Q}^{q \times 2n}$, $b \in \mathbb{Q}^p$, $c \in \mathbb{Q}^q$ for some $p, q > 0$.

- the *update* constraint is *deterministic* if for each $\mathbf{x}$ satisfying the *guard* constraint there is at most one $\mathbf{x}'$

Single-path loop

A loop can be interpreted as a state–transition model with transition $x = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \in \mathbb{Q}^{2n}$ from state $\mathbf{x}$ to state $\mathbf{x}'$. The set of all transitions $\mathbf{x}''$ is known as *transition polyhedron* $\mathcal{Q}$.

Transition polyhedron

The transition polyhedron $\mathcal{Q}$ is determined by the linear constraint

$$A''\mathbf{x}'' \leq \mathbf{c}''$$

where

$$A'' = \begin{pmatrix} B & 0 \\ & A \end{pmatrix}, \; \mathbf{c}'' = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

Multipath loop

Multipath linear-constraint loop

An *MLC* loop over variables $x_1, \ldots, x_n$ is given in the form

$$\vee_{i=1}^{k} \text{while}(B_i\mathbf{x} \leq \mathbf{b}_i) \text{ do } A_i \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \leq \mathbf{c}_i$$

The loop is specified by transition polyhedra $\mathcal{Q}_i$, and in each iteration a polyhedron $\mathcal{Q}_i$ might is chosen non-deterministically.

Linear Ranking Function

Linear Ranking Functions

A linear function $\rho : \mathbb{Q}^n \to \mathbb{Q}$

$$\rho(\mathbf{x}) \; = \; \vec{\lambda} \cdot \mathbf{x} + \lambda_0$$

where $\vec{\lambda} \in \mathbb{Q}^n, \lambda_0 \in \mathbb{Q}$, is linear ranking function for MLC loop iff the following holds for any element $\begin{pmatrix} \mathbf{x} \\ \mathbf{x}'' \end{pmatrix} \in \mathcal{Q}_i$

$$\rho(\mathbf{x}) \; \geq 0 \tag{1}$$

$$\rho(\mathbf{x}) - \rho(\mathbf{x}') \; \geq 1 \tag{2}$$

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Decision problem

LinRF

Given an MLC loop, does there exist a LRF for this loop?

- $\textsc{LinRF}(\mathbb{Q})$ is PTIME-decidable
- $\textsc{LinRF}(\mathbb{Z})$ is $\textsc{coNP-complete}$

  The proof is designed as follows:

  1. Show that $\textsc{LinRF}(\mathbb{Z})$ is strongly coNP-hard.
  2. Show that $\textsc{LinRF}(\mathbb{Z})$ is in coNP by presenting a class of polynomially-checkable functions verifying *nonexistence* of a LRF.

Hardness

Definition

A problem $\mathbf{X}$ is coNP-hard if for every problem $\mathbf{Y} \in$ coNP, $\mathbf{Y}$ is polynomially-reducible to $\mathbf{X}$, denoted as $\mathbf{Y} \leq_P \mathbf{X}$.

A hardness of a problem Z is proven by reducing a known coNP-hard problem to X, since $\forall Y \in$ coNP, $\; Y \leq_P X \leq_P Z$.
Here, we use the Karp's 0-1 Integer Linear Programming problem which is known to be coNP-complete.

Definition

Is the polyhedron given by $B\mathbf{x} \leq \mathbf{b}$ empty?

Hardness

Given the polyhedron defined by $B \in \mathbb{Z}^{m \times n}, \mathbf{b} \in \mathbb{Q}^m$, consider a SLC loop

$$\text{while} \begin{pmatrix} B - \mathcal{I} \\ 0 - \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix} \leq \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix} \text{do} \begin{pmatrix} \mathbf{x}' \\ \mathbf{z}' \end{pmatrix} = \begin{pmatrix} \mathbf{x}' \\ \mathbf{0} \end{pmatrix}$$

1. The system $B\mathbf{x} \leq \mathbf{b}$ has an integer solution $\mathbf{x}$.
   Loop conditions are satisfied for the starting point consisting of the solution $\mathbf{x}$ and $\mathbf{z} = \mathbf{0}$. The update does not update variables, the loop does not terminate and there can be no LRF.
2. The system $B\mathbf{x} \leq \mathbf{b}$ has no integer solution.
   For the initial state it must hold that $z_1 + \cdots + z_m > 0$ since the guard condition $B\mathbf{x} - \mathcal{I}\mathbf{z} \leq \mathbf{b}$ must have a solution. The loop terminates in the next iteration and $z_1 + \cdots + z_m$ is an LRF.

Solving $\text{LinRF}(\mathbb{Z})$ determines whether there is no integer solution for $B\mathbf{x} \leq \mathbf{z}$. Therefore, an coNP-hard problem can be solved by solving $\text{LinRF}(\mathbb{Z})$ and $\text{LinRF}(\mathbb{Z})$ must be coNP-hard.

coNP proof

To prove that $\textsc{LinRF}(\mathbb{Z}) \in$ coNP, the authors show that the complement problem of nonexistence of LRF is in NP.

- First, prove that existence of witnesses satisfying certain criteria implies nonexistence of LRF.
- Later, prove that there is a witness with bit-size polynomial in the input bit-size.

Witnesses

Witness

The transition $\mathbf{x}'' = \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} \in I(\mathcal{Q})$ is a witness against a candidate LRF function $(\vec{\lambda}, \lambda_0)$ if one of conditions below doesn't hold

$$\rho(\mathbf{x}) \geq 0 \tag{3}$$

$$\rho(\mathbf{x}) - \rho(\mathbf{x}') \geq 1 \tag{4}$$

$W(\mathbf{x}'')$ denotes a set of all functions witnessed against by $\mathbf{x}''$.

Witnesses

H-Witness

The transition $\mathbf{y}'' = \begin{pmatrix} \mathbf{y} \\ \mathbf{y}' \end{pmatrix} \in \mathcal{R}_{\mathcal{Q}}$ is a witness against a candidate LRF function $(\vec{\lambda}, \lambda_0)$ if one of conditions below doesn't hold.

$$\vec{\lambda}\mathbf{y} \geq 0 \tag{5}$$

$$\vec{\lambda}(\mathbf{y} - \mathbf{y}') \geq 1 \tag{6}$$

$W_H(\mathbf{y}'')$ denotes a set of all functions witnessed against by $\mathbf{y}''$.

Interpretation. It can be shown that for each point $\mathbf{y}''$, there is an integer point $\mathbf{z}'' \in I(\mathcal{Q})$ such that $(\vec{\lambda}, \lambda_0)$ witnessed against by $\mathbf{y}''$ can't be LRF.

Witnessing lack of LRF

Theorem

Let $X \subseteq I(\mathcal{Q})$, $Y \subseteq I(\mathcal{R}_P)$ and $WS(X, Y) = \bigcup_{\mathbf{x}'' \in X} W(\mathbf{x}'') \cup \bigcup_{\mathbf{y}'' \in Y} W_H(\mathbf{y}'')$.

If $WS(X, Y) = \mathbb{Q}^{n+1}$, then there is no LRF for $I(\mathbb{Q})$.

How to prove it?

1. For every function $(\vec{\lambda}, \lambda_0) \in W(\mathbf{x}'')$ for some $\mathbf{x}''$, the function can't be LRF since it doesn't fulfill basic conditions.
2. For every function $(\vec{\lambda}, \lambda_0) \in W_H(\mathbf{y}'')$ for some $\mathbf{y}''$, show that conditions for an H-Witness imply that that there is $\mathbf{z}'' \in I(\mathcal{Q})$ such that $(\vec{\lambda}, \lambda_0)$ can't be LRF.

Witness existence

Theorem

For a transition polyhedron $\mathcal{Q}$, let the integer hull $\mathcal{Q}_I$ be represented as $\text{convhull}(x''_1, \ldots, x''_m) + \text{cone}(y''_1, \ldots, y''_t)$. If there is no LRF for $I(\mathcal{Q})$, then $WS(\{x''_1, \ldots, x''_m\}, \{y''_1, \ldots, y''_t\}) = \mathbb{Q}^{n+1}$.

How to prove it?
Assume the contra-positive $WS(x''_1, \ldots, x''_m, y''_1, \ldots, y''_t) \neq \mathbb{Q}^{n+1}$ and show that there exists $(\vec{\lambda}, \lambda_0) \in \mathbb{Q}^{n+1}$ such that $(\vec{\lambda}, \lambda_0)$ is a LRF for $I(\mathcal{Q})$.

Witness summary

- Existence of X and Y s.t. $WS(X, Y) = \mathbb{Q}^{n+1}$ implies lack of existence of a LRF.
- Lack of existence of LRF implies that there are sets X and Y such that $WS(X, Y) = \mathbb{Q}^{n+1}$.

There is no LRF for $I(\mathcal{Q})$ if and only if there are two sets $X \subseteq I(\mathcal{Q})$ and $Y \subseteq I(\mathcal{R}_P)$ such that $WS(X, Y) = \mathbb{Q}^{n+1}$

The condition $WS(X, Y) = \mathcal{Q}^{n+1}$ is equivalent to stating the system Ψ_{WY} of inequalities over $n + 1$ variables $\lambda_0, \ldots, \lambda_n$, created as a conjunction of all witness and H-witness conditions, has no rational solution.

Witness size

Theorem

If there exists a witness for the nonexistence of LRF for I(Q), there exists one with $X \subseteq I(\mathcal{Q})$ and $Y \subseteq I(\mathcal{R}_{\mathcal{Q}})$ such that $|X \cup Y| \leq n + 1$ and the bit-size of $X \cup Y$ is polynomial in the bit-size of the input.

How to prove that?

- A corollary to Farkas' lemma shows that if there is no solution to Ψ_{WY} over $n + 1$ variables, one can select at most $n + 1$ inequalities . In this case, these inequalities describe witness conditions and it is sufficient to describe at most $n + 1$ points from either X or Y. Thus, $X \cup Y$ contains these points and $|X \cup Y| \leq n + 1$.
- The bit-size of a witness is bounded by $96n^4 \left\| \mathcal{Q}_{\mathcal{I}} \right\|_b$

Summary

Theorem

$\textsc{LinRF}(\mathbb{Z}) \in$ coNP for SLC loops.

The witness for nonexistence of LRF consists of $n+1$ integer points from $X \cup Y$.

- Its bit-size is polynomially bounded by the bit-size of polyhedron $\mathcal{Q}_I$, as shown before.
- Verification is defined as follows
    1. For each element of sets X and Y verify that either $A''\mathbf{x}'' \leq c''$ or $A''\mathbf{y}'' \leq 0$ holds, respectively.
    2. Construct the rational system of inequalities Ψ_{XY} and verify that there are no solutions to it. Since it is a Linear Programming problem, it can be done in polynomial time.

Multipath loops

Theorem

$\textsc{LinRF}(\mathbb{Z}) \in$ coNP for MLC loops.

The proof follows the same logic as previous one, with extensions such that

- $(\vec{\lambda}, \lambda_0)$ must be LRF for each $I(\mathcal{Q}_i)$. Existence of LRF for each path does not guarantee existence of LRF for MLC loop since the intersection of all possible LRFs might be empty.
- Witness sets X and Y are constructed from witness sets for X_i and Y_i for each transition polyhedra.

LRF synthesis

Given MLC loops defined by transition polyhedra $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$.

1. Compute the generator representation for each $\mathcal{Q}_{iI}$

$$\mathcal{Q}_{iI} = \text{convhull}(X_i) + \text{cone}(Y_i)$$

2. Define $X = X_i \cup \cdots \cup X_k$ and Y in a similar manner.
3. Construct a system of linear inequalities $\Psi_{WS}(X, Y)$
4. Find a Linear Programming solution $(\vec{\lambda}, \lambda_0)$ to $\Psi_{WS}(X, Y)$.

There might be exponential number of generators for a convex hull. If there is a LRF, its bit size is polynomial in $\max_i \|\mathcal{Q}_i\|_b$.

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Integral polyhedron

Theorem

Let $\mathcal{Q}$ be a transition polyhedron of a given SLC loop, and let ρ be a linear function. If $\mathcal{Q}$ is integral, then ρ is a LRF for $\mathcal{Q}$ iff ρ is a LRF for $\mathcal{I}(\mathcal{Q})$.

How to prove that?

- $\Longrightarrow$

  If ρ is LRF for $\mathcal{Q}$, it has to be LRF for $I(\mathcal{Q})$ since $I(\mathcal{Q}) \subseteq \mathcal{Q}$

- $\Longleftarrow$

  Every rational point of an integer polyhedron $\mathcal{Q}$ is a convex combination of integer points $\mathbf{x}'' = \sum a_i \mathbf{x}_i''$ where $\sum a_i = 1$.

$$\rho(\mathbf{x}) = \sum (a_i \vec{\lambda} \mathbf{x}_i + \lambda_0) = \sum a_i \rho(\mathbf{x}_i) \geq 0$$

$$\rho(\mathbf{x}) - \rho(\mathbf{x}') = \sum a_i \vec{\lambda}(\mathbf{x}_i - \mathbf{x}_i') = \sum a_i \rho(\mathbf{x}_i) - \rho(\mathbf{x}_\mathbf{i}') \geq 1$$

Integral polyhedron

Theorem

Let $\mathcal{Q}$ be a transition polyhedron of a given SLC loop, and let ρ be a linear function. If $\mathcal{Q}$ is integral, then ρ is a LRF for $\mathcal{Q}$ iff ρ is a LRF for $\mathcal{I}(\mathcal{Q})$.

When can we solve the $\textsc{LinRF}(\mathbb{Z})$ problem in polynomial time?

- transition polyhedron $\mathcal{Q}$ is integral
- the integer hull of polyhedra $\mathcal{Q}$ can be computed in polynomial time
- the condition polyhedron C is integral or computable in polynomial time and the update $\mathbf{x}' = A'\mathbf{x} + \mathbf{c}'$ has integral coefficients

  Proof. Since C is integral, $\mathbf{x}$ is a convex combination of integer points $\mathbf{x} = \sum a_i \mathbf{x}_i$. Thus, $\mathbf{x}' = \sum a_i(A'\mathbf{x}_i + \mathbf{c}')$. Therefore, for an arbitrary point $\mathbf{x}'' \in \mathcal{Q}$

  $$\mathbf{x}'' = \sum a_i \begin{pmatrix} x_i \\ A'\mathbf{x}_i' + c_i \end{pmatrix}$$

  Which is a convex combination of integer points from $I(\mathcal{Q})$ and $\mathcal{Q}$ is integral.

Integral polyhedron

When can we solve the $\textsc{LinRF}(\mathbb{Z})$ problem in polynomial time?

- SLC loops of form while$(B\mathbf{x} \leq 0)$ do $\mathbf{x}' = A'\mathbf{x} + \mathbf{c}'$
  A cone is integer polyhedron since it can always have generators with integral coefficients.
- SLC loops where transition polyhedron $\mathcal{Q}$ is totally unimodular.
- SLC loops where condition polyhedron $\mathcal{C}$ is totally unimodular and the update is linear with integer coefficients.
  The polyhedron $P = \{\mathbf{x} | A\mathbf{x} + \mathbf{b}\}$ is totally unimodular if $\mathbf{b}$ is integer and A is totally unimodular. Such polyhedron is always integral.

Two-variable inequalities

TVPI

A two-variable per inequality constraint is of the form

$$ax + by \leq d$$

where $a, b, d \in \mathbb{Q}$.

For a two-dimensional polyhedra defined by m such constraints, the integer hull may be computed with Harvey's method in $\mathcal{O}(m \log A_{max})$.

PTVPI

A product of independent two-dimensional TVPI constraints is a set T which can be partitioned into $T_1, \ldots, T_n$ such that

1. T_i is two-dimensional
2. each pair T_i and T_j do not share variables

Two-variable inequalities

Integer hull

The integer hull of polyhedron $\mathcal{T}$ given by PTVPI constraints can be computed in polynomial time.

This is proven by computing the integer hull for each subset T_i, which can be done in polynomial time with Harvey's method. The product of subsets T_{il} and T_{jl} is integral since each face of product has an integer point coming from faces of T_{il} and T_{jl}. Thus, the final product is integral and contains all integer points of $\mathcal{T}$.

We can we solve the $\textsc{LinRF}(\mathbb{Z})$ problem in polynomial time for

- SLC loops where transition polyhedron $\mathcal{Q}$ is PTVPI.
- SLC loops where condition polyhedron $\mathcal{C}$ is PTVPI and the update is linear with integer coefficients.

Octagonal polyhedron

Octagonal relation

A TVPI constraint with coefficients a, b from $\{0, \pm 1\}$ is octagonal.

- Computing a tight closure infers new inequalities and replaces d with $\lfloor d \rfloor$. Tightening eliminates non-integer points and might create an integer hull.
- Computing the integer hull for octagonal polyhedron can't have a polynomial-time algorithm.

Proof. The proof uses a polyhedron defined by inequalities for a graph K_n. Such polyhedron must have an exponential number of facets and its integer hull must have the same number of inequalities. A general set of octagonal inequalities can be shown to be equivalent to such system which proves that computing integer hull add exponential number of inequalities.

Octagonal polyhedron

Can there be a polynomial time algorithm for $\textsc{LinRF}(\mathbb{Z})$ defined with octagonal polyhedron?

- Might be for a transition polyhedron $\mathcal{Q}$ which is octagonal.
- Not when the guard polyhedron $\mathcal{Q}$ is octagonal since then the problem is strongly coNP-hard, by a reduction from 3SAT to a complement of $\textsc{LinRF}(\mathbb{Z})$.

Strongly polynomial

The problem is solved by a strongly polynomial algorithm if the number of operations is polynomial in the size of input problem and arithmetical operations are performed on numbers bounded by polynomial in the input bit-size. The $\textsc{LinRF}(\mathbb{Z})$ is decidable in strongly polynomial time for SLC loop $A''\mathbf{x}'' \leq \mathbf{c}''$ when entries of A'' are from $\{0, \pm 1\}$ and

- The transition polyhedron is integral.
  One can use a strongly polynomial time algorithm for $\textsc{LinRF}\mathbb{Q}$.
- The transition polyhedron is PTVPI or condition polyhedron PTVPI and the update is linear with integer coefficients.
  The Harvey's algorithm is strongly polynomial in such case and constraints created when computing the integer hull procedure have coefficients in $\{0, \pm 1\}$ and it adds at most $\mathcal{O}(m \log A_{max})$ inequalities.

MLC Loop

The $\textsc{LinRF}(\mathbb{Z})$ is decidable in polynomial time for MLC loops, when for path it holds that

- The transition polyhedron integral
- The condition polyhedron is integral or its integer hull can be computed in polynomial and the update is linear with integer coefficients.
- The transition polyhedron is PTVPI
- The condition polyhedron is PTVPI and the update is linear with integer coefficients.

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Summary

- Unless $\mathrm{P} = \mathrm{NP}$, there is no polynomial time algorithm for the general $\textsc{LinRF}(\mathbb{Z})$ problem.
- The problem can be decided in polynomial time in several special cases since $\textsc{LinRF}(\mathbb{Z})$ might be equivalent to $\textsc{LinRF}(\mathbb{Q})$.
- The existence of LRF proves that loop terminates but the reversed statement does not always hold.
- The paper proposes two algorithms for finding LRF:
  1. Find a generator representation for the integer hull and solve an appropriate set of rational inequalities.
  2. Compute the integer hull for transition polyhedra and apply *Podelski-Rybalchenko procedure*.
- The computation of integer hull might be simplified by decomposing inequalities into independent sets. There are polynomial-time algorithms for TVPI polyhedron and octogonal polyhedron, which is not complete.